Functional codes arising from quadric intersections with Hermitian varieties

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Abstract

We investigate the functional code $C_h(\mathrm{X})$ introduced by G. Lachaud [10] in the special case where X is a non-singular Hermitian variety in $\mathrm{PG}(N, q^2)$ and $h = 2$. In [4], F. Edoukou solved the conjecture of Sørensen [11] on the minimum distance of this code for a Hermitian variety X in $\mathrm{PG}(3, q^2)$. In this paper, we will answer the question about the minimum distance in general dimension N, with $N < O(q^2)$. We also prove that the small weight codewords correspond to the intersection of X with the union of 2 hyperplanes.

1 Introduction

We study the functional code $C_2(\mathrm{X})$ in $\mathrm{PG}(N, q^2)$, where X is a non-singular Hermitian variety $\mathrm{H}(N, q^2)$. Let $\mathrm{X} = \{P_1, \ldots, P_n\}$, where we normalize the coordinates of these points with respect to the leftmost non-zero coordinate. Let $\mathcal{F}$ be the set of all homogeneous quadratic polynomials $f(X_0, \ldots, X_N)$ defined by $N + 1$ variables with coefficients in $\mathbb{F}_{q^2}$. The functional code $C_2(\mathrm{X})$ is the linear code

$$C_2(\mathrm{X}) = \{(f(P_1), \ldots, f(P_n)) || f \in \mathcal{F} \cup \{0\}\}.$$

This linear code has length $n = |\mathrm{X}|$ and dimension $k = \binom{N+2}{2}$ over $\mathbb{F}_{q^2}$. The third fundamental parameter of this linear code is its minimum distance d. Since the code is linear, this minimum distance corresponds to the minimum weight of the code. The small weight codewords, i.e., the codewords having the minimum weight or a weight close to the minimum weight, arise from the quadrics having the (almost) largest intersections with X.

Sørensen [11] conjectured that the maximum size for the intersection of a quadric Q with the Hermitian variety $\mathrm{H}(3, q^2)$ in $\mathrm{PG}(3, q^2)$ is equal to $2q^3 + 2q^2 - q + 1$. The correctness of this conjecture was proven by Edoukou in [3].

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More precisely, Edoukou not only proved that the maximum size for the intersection of a quadric Q with the Hermitian variety $\mathrm{H}(3,q^2)$ in $\mathrm{PG}(3,q^2)$ is equal to $2q^3+2q^2-q+1$; he also proved that the second largest intersection size of a quadric Q with the Hermitian variety $\mathrm{H}(3,q^2)$ in $\mathrm{PG}(3,q^2)$ is at most $2q^3+q^2+1$.

Regarding the largest intersection sizes of a quadric Q with the Hermitian variety $\mathrm{H}(4,q^2)$ in $\mathrm{PG}(4,q^2)$, Edoukou [5] determined the five largest intersection sizes, leading to the 5 smallest weights for the code $C_2(\mathrm{X})$, $\mathrm{X}=\mathrm{H}(4,q^2)$.

In [5, Conjecture 2, p. 145], he also stated that the five smallest weights for the code $C_2(\mathrm{X})$, $\mathrm{X}=\mathrm{H}(N,q^2)$, arise from the intersections of X with the quadrics which are the union of two distinct hyperplanes.

We determine the 5 smallest weights of $C_2(\mathrm{X})$, $\mathrm{X}=\mathrm{H}(N,q^2)$, $N<O(q^2)$, via geometrical arguments, and prove the validity of the conjecture of Edoukou for $N<O(q^2)$. These 5 smallest weights will be the small weights of the code $C_2(\mathrm{X})$, $\mathrm{X}=\mathrm{H}(N,q^2)$, on which we will concentrate.

First of all, we will investigate the different intersections of quadrics Q in $\mathrm{PG}(4,q^2)$ with $\mathrm{H}(4,q^2)$; leading to a lower bound on the intersection size guaranteeing that any quadric having more than this number of points in common with $\mathrm{H}(4,q^2)$ must be the union of two hyperplanes. We use this result to find a bound on the intersection sizes of absolutely irreducible quadrics with the non-singular Hermitian variety $\mathrm{H}(N,q^2)$. Here this lower bound on the intersection size guarantees that Q is the union of 2 hyperplanes. Using this bound, we prove that the small weight codewords correspond to quadrics which are the union of 2 hyperplanes. There are several possibilities for the intersection of such a quadric with a non-singular Hermitian variety X. So we can construct tables with the 5 smallest weights of the functional code $C_2(\mathrm{X})$, X a non-singular Hermitian variety in $\mathrm{PG}(N,q^2)$, $N<O(q^2)$.

The results of this article continue the research on the small weight codewords of functional codes performed in [6, 7]. In [6], we determined the smallest weights of the non-zero codewords of the functional codes $C_2(\mathrm{Q})$, which are defined by the intersections of all quadrics with a non-singular quadric Q in $\mathrm{PG}(N,q)$, and in [7], we determined the smallest weights of the non-zero codewords of the functional codes $C_{herm}(\mathrm{X})$, which are defined by the intersections of all Hermitian varieties with a non-singular Hermitian variety in $\mathrm{PG}(N,q^2)$. In these cases, the smallest weight codewords arise in [6] from the intersections of Q with the quadrics which are the union of two hyperplanes, and in [7] from the intersections of X with the Hermitian varieties which are the union of $q+1$ hyperplanes through a common $(N-2)$-dimensional space of $\mathrm{PG}(N,q^2)$.

In the article [6], the crucial element was the fact that the intersection V of two quadrics Q and Q$'$ lies in all the $q+1$ quadrics $\lambda\mathrm{Q}+\mu\mathrm{Q}'$, $(\lambda,\mu)\in\mathbb{F}_q^2\setminus\{(0,0)\}$, of the pencil of quadrics defined by Q and Q$'$ and similarly for the second article [7], the crucial element was the fact that the intersection V of two Hermitian varieties X and X$'$ in $\mathrm{PG}(N,q^2)$ lies in all the $q+1$ Hermitian varieties $\lambda\mathrm{X}+\mu\mathrm{X}'$, $(\lambda,\mu)\in\mathbb{F}_q^2\setminus\{(0,0)\}$, of the pencil of Hermitian varieties defined by X and X$'$. This enabled us to obtain results for general dimensions N.

We cannot use this fact in this article. A quadric and a Hermitian variety do not define together a pencil of quadrics or of Hermitian varieties. This implied that different arguments had to be used, which enabled us to obtain results up to dimension $N < O(q^2)$ for the Hermitian variety X in $\mathrm{PG}(N, q^2)$.

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2 Quadrics and Hermitian varieties

By π_i, we denote a projective subspace of dimension i in $\mathrm{PG}(N, q^2)$. We will often use the term *space* instead of *projective subspace*. The space generated by two spaces π_i and $\pi_{i'}$ is denoted by $\langle \pi_i, \pi_{i'} \rangle$.

For the fundamental properties of quadrics and Hermitian varieties, we refer to [9, Chapters 22 and 23]. We repeat the relevant properties for the arguments in this article.

The non-singular quadrics in $\mathrm{PG}(N, q^2)$ are equal to:

- the *non-singular parabolic quadrics* $\mathrm{Q}(N, q^2)$ in $\mathrm{PG}(N = 2N', q^2)$ having standard equation $X_0^2 + X_1X_2 + \cdots + X_{2N'-1}X_{2N'} = 0$. These quadrics contain $q^{4N'-2} + \cdots + q^2 + 1$ points, and the largest dimensional spaces contained in a non-singular parabolic quadric of $\mathrm{PG}(2N', q^2)$ have dimension $N' - 1$,
- the *non-singular hyperbolic quadrics* $\mathrm{Q}^+(N, q^2)$ in $\mathrm{PG}(N = 2N' + 1, q^2)$ having standard equation $X_0X_1 + \cdots + X_{2N'}X_{2N'+1} = 0$. These quadrics contain $(q^{2N'} + 1)(q^{2N'+2} - 1)/(q^2 - 1) = q^{4N'} + q^{4N'-2} + \cdots + q^{2N'+2} + 2q^{2N'} + q^{2N'-2} + \cdots + q^2 + 1$ points, and the largest dimensional spaces contained in a non-singular hyperbolic quadric of $\mathrm{PG}(N = 2N' + 1, q^2)$ have dimension N',
- the *non-singular elliptic quadrics* $\mathrm{Q}^-(N, q^2)$ in $\mathrm{PG}(N = 2N'+1, q^2)$ having standard equation $f(X_0, X_1)+X_2X_3+\cdots+X_{2N'}X_{2N'+1} = 0$, where $f(X_0, X_1)$ is an irreducible quadratic polynomial over $\mathbb{F}_{q^2}$. These quadrics contain $(q^{2N'+2} + 1)(q^{2N'} - 1)/(q^2 - 1) = q^{4N'} + q^{4N'-2} + \cdots + q^{2N'+2} + q^{2N'-2} + \cdots + q^2 + 1$ points, and the largest dimensional spaces contained in a non-singular elliptic quadric of $\mathrm{PG}(2N' + 1, q^2)$ have dimension $N' - 1$.

The non-singular Hermitian variety $\mathrm{H}(N, q^2)$ in $\mathrm{PG}(N, q^2)$ has standard equation $X_0^{q+1} + X_1^{q+1} + \cdots + X_N^{q+1} = 0$. This variety contains $\frac{(q^{N+1}+(-1)^N)(q^N+(-1)^{N+1})}{q^2-1}$ points, and the largest dimensional spaces contained in a non-singular Hermitian variety of $\mathrm{PG}(N, q^2)$ have dimension $\lfloor \frac{N-1}{2} \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x.

All the quadrics and Hermitian varieties of $\mathrm{PG}(N, q^2)$, including the non-singular ones, can be described as a quadric/Hermitian variety having an s-dimensional *vertex* π_s of singular points, $s \geq -1$, and having a non-singular *base* X_{N-s-1} in an $(N - s - 1)$-dimensional space skew to π_s. We denote such a quadric or Hermitian variety in $\mathrm{PG}(N, q^2)$ with vertex π_s and base X_{N-s-1} by $\pi_s\mathrm{X}_{N-s-1}$. The points P of the vertex π_s of a quadric or

Hermitian variety $\pi_s X_{N-s-1}$ are called the *singular* points of $\pi_s X_{N-s-1}$, while the points of $\pi_s X_{N-s-1} \setminus \pi_s$ are called *non-singular*. A quadric or Hermitian variety $\pi_s X_{N-s-1}$ is called *singular* when it has a vertex π_s of dimension $s \geq 0$.

A line intersecting the quadric or Hermitian variety X in a unique point is called a *tangent line*. A *tangent hyperplane* through a point $P \in$ X is a hyperplane such that all lines through P in this hyperplane are either tangent lines or either contained in X. Such a hyperplane is denoted by $T_P(\mathrm{X})$. A non-singular point of a quadric or Hermitian variety X has a unique tangent hyperplane; for a singular point P of X, every hyperplane through P is a tangent hyperplane to X.

Consider a non-singular quadric or Hermitian variety X in N dimensions, then a non-tangent hyperplane intersects X in a non-singular quadric or non-singular Hermitian variety, and a tangent hyperplane intersects this non-singular quadric or Hermitian variety X in a cone $\pi_0 \mathrm{X}'$, with X' a quadric or Hermitian variety in $N-2$ dimensions of the same type as X; see [1, 2] for these properties in the case of Hermitian varieties.

We call the largest dimensional spaces contained in a quadric or Hermitian variety the *generators* of this quadric or Hermitian variety.

The quadrics having the largest size are the union of two distinct hyperplanes of $\mathrm{PG}(N, q^2)$, and have size $2q^{2N-2} + q^{2N-4} + \cdots + q^2 + 1$.

As we mentioned in the introduction, the smallest weight codewords of the code $C_2(\mathrm{X})$ correspond to the quadrics Q having the largest intersections with the Hermitian variety X of $\mathrm{PG}(N, q^2)$. We will show that the largest intersections arise from the quadrics Q that are the union of two distinct hyperplanes of $\mathrm{PG}(N, q^2)$, when $N < O(q^2)$. This proves the conjecture of F.A.B. Edoukou [5] in small dimensions N.

Finally, the set of $q+1$ transversals of three pairwise skew lines in $\mathrm{PG}(3, q)$ is called a *regulus*. Three lines of a regulus define again a regulus, called the *opposite regulus*. A hyperbolic quadric $\mathrm{Q}^+(3, q)$ is a pair of complementary reguli.

3 Dimension 4

The goal is to look for a bound W_4 on the intersection size of an absolutely irreducible quadric Q with the Hermitian variety X (=$\mathrm{H}(4, q^2)$), in such a way that if the intersection size of Q $\cap$ X is larger than this bound, then the quadric Q has to be the union of 2 hyperplanes. Therefore we search for the largest intersection size of an absolutely irreducible quadric with X. This problem was first investigated by Edoukou [5]. We present here an alternative approach.

3.1 The quadric Q is the non-singular quadric $\mathrm{Q}(4, q^2)$

Lemma 3.1 *If* $\mathrm{Q}^+(3, q^2) \cap \mathrm{H}(3, q^2)$ *contains 3 skew lines, then the intersection consists of* $2(q+1)$ *lines forming a hyperbolic quadric* $Q^+(3, q)$ *and* $|\mathrm{Q}^+(3, q^2) \cap \mathrm{H}(3, q^2)| = 2q^3 + q^2 + 1$.

Proof. This is [8, Lemma 19.3.1].

This implies that

$$\begin{aligned} |\mathrm{Q}^+(3,q^2)\cap \mathrm{H}(3,q^2)| &= (q+1)(q^2+1)+(q^2-q)(q+1) \\ &= 2q^3+q^2+1. \end{aligned}$$

□

Lemma 3.2 *If* $\mathrm{Q}^+(3,q^2)\cap \mathrm{H}(3,q^2)$ *contains at most 2 skew lines, then* $|\mathrm{Q}^+(3,q^2)\cap \mathrm{H}(3,q^2)| \leqslant q^3+3q^2-q+1$.

Proof. (see also [3]) We count according to the lines of one regulus of $\mathrm{Q}^+(3,q^2)$:

$$\begin{aligned} |\mathrm{Q}^+(3,q^2)\cap \mathrm{H}(3,q^2)| &\leqslant 2(q^2+1)+(q^2-1)(q+1) \\ &\leqslant q^3+3q^2-q+1. \end{aligned}$$

□

Lemma 3.3 *Let* L *be a line of* $\mathrm{Q}(4,q^2)$ *containing at most* q *points of* $\mathrm{Q}(4,q^2)\cap\mathrm{H}(4,q^2)$, *then* $|\mathrm{Q}(4,q^2)\cap \mathrm{H}(4,q^2)| \leqslant q^5+3q^4+2q^2+q+1$.

Proof. Let $P\in L$ with $P\notin \mathrm{Q}(4,q^2)\cap\mathrm{H}(4,q^2)$. Take a line M of $\mathrm{Q}(4,q^2)$ intersecting L in P. Consider the plane $\langle L,M\rangle$. Then $\langle L,M\rangle$ lies in the tangent hyperplane $T_P(\mathrm{Q}(4,q^2))$ to $\mathrm{Q}(4,q^2)$ and on q^2 3-dimensional spaces sharing a hyperbolic quadric $\mathrm{Q}^+(3,q^2)$ with $\mathrm{Q}(4,q^2)$. No $\mathrm{Q}^+(3,q^2)$ can intersect $\mathrm{H}(4,q^2)$ in $q+1$ lines of both reguli, since L has only q points of the intersection $\mathrm{Q}(4,q^2)\cap\mathrm{H}(4,q^2)$. So $|\mathrm{Q}(4,q^2)\cap\mathrm{H}(4,q^2)| \leqslant q^2(q^3+3q^2-q+1)+|T_P(\mathrm{Q}(4,q^2))\cap\mathrm{Q}(4,q^2)\cap\mathrm{H}(4,q^2)|$.
If $P\notin \mathrm{Q}(4,q^2)\cap\mathrm{H}(4,q^2)$, then $|T_P(\mathrm{Q}(4,q^2))\cap\mathrm{Q}(4,q^2)\cap\mathrm{H}(4,q^2)| \leqslant (q+1)(q^2+1)$.
So $|\mathrm{Q}(4,q^2)\cap\mathrm{H}(4,q^2)| \leqslant q^5+3q^4+2q^2+q+1$. □

Remark 3.4 *From now on, we assume that every line of* $\mathrm{Q}(4,q^2)$ *shares at least* $q+1$ *points with* $\mathrm{H}(4,q^2)$. *So all lines of* $\mathrm{Q}(4,q^2)$ *share* $q+1$ *or* q^2+1 *points with* $\mathrm{H}(4,q^2)$, *since a line having more than* $q+1$ *points of* $\mathrm{H}(4,q^2)$ *is contained in* $\mathrm{H}(4,q^2)$.

Lemma 3.5 *Let* $P\in \mathrm{Q}(4,q^2)\cap\mathrm{H}(4,q^2)$, *then* $T_P(\mathrm{Q}(4,q^2))\neq T_P(\mathrm{H}(4,q^2))$.

Proof. Assume that $T_P(\mathrm{Q}(4,q^2)) = T_P(\mathrm{H}(4,q^2))$. Let $\mathrm{Q}(2,q^2)$ be the base of $T_P(\mathrm{Q}(4,q^2))\cap \mathrm{Q}(4,q^2)$ and let $\mathrm{H}(2,q^2)$ be the base of $T_P(\mathrm{H}(4,q^2))\cap\mathrm{H}(4,q^2)$. Take a line L through P to a point of $\mathrm{Q}(2,q^2)\backslash\mathrm{H}(2,q^2)$. This line L only shares P with $\mathrm{H}(4,q^2)$, while it should contain at least $q+1$ points of $\mathrm{H}(4,q^2)$. □

Lemma 3.6 *Assume that all lines of* $\mathrm{Q}(4,q^2)$ *share* $q+1$ *or* q^2+1 *points with* $\mathrm{H}(4,q^2)$, *then* $|\mathrm{Q}(4,q^2)\cap\mathrm{H}(4,q^2)| \leqslant q^5+3q^4-4q^2+3q+1$.

Proof. Let P be a point of $\mathrm{Q}(4, q^2)$ not lying in the intersection $\mathrm{Q}(4, q^2) \cap \mathrm{H}(4, q^2)$, and take 2 lines L and M of $\mathrm{Q}(4, q^2)$ through P. All $q^2 + 1$ lines of $\mathrm{Q}(4, q^2)$ through P contain $q + 1$ points of $\mathrm{Q}(4, q^2) \cap \mathrm{H}(4, q^2)$, so $|T_P(\mathrm{Q}(4, q^2)) \cap \mathrm{Q}(4, q^2) \cap \mathrm{H}(4, q^2)| = (q + 1)(q^2 + 1)$.

Consider the $q + 1$ points $P_1, \ldots, P_{q+1}$ of $L \cap \mathrm{Q}(4, q^2) \cap \mathrm{H}(4, q^2)$. They lie on at most 2 lines contained in $\mathrm{Q}(4, q^2) \cap \mathrm{H}(4, q^2)$ (Lemma 3.5). For, such a line through a point P_i lies in the tangent hyperplanes $T_P(\mathrm{Q}(4, q^2))$ and $T_P(\mathrm{H}(4, q^2))$. But these tangent hyperplanes only have a plane in common and this plane has at most two lines through P_i contained in $\mathrm{Q}(4, q^2) \cap \mathrm{H}(4, q^2)$. So at most two of the q^2 distinct hyperbolic quadrics $\mathrm{Q}^+(3, q^2)$ of $\mathrm{Q}(4, q^2)$ through $\langle L, M\rangle$ can intersect $\mathrm{H}(4, q^2)$ in $2(q + 1)$ lines, so we get at most twice $2q^3 + q^2 + 1 - 2(q + 1) = 2q^3 + q^2 - 2q - 1$ extra intersection points. At least $q^2 - 2$ times, we get at most $q^3 + 3q^2 - q + 1 - 2(q + 1) = q^3 + 3q^2 - 3q - 1$ extra intersection points.

So in total there are at most $q^5 + 3q^4 - 4q^2 + 3q + 1$ intersection points. $\square$

3.2 The quadric cone $\mathrm{Q} = \pi_0\mathrm{Q}^-(3, q^2)$

Case I: $\mathrm{H}(4, q^2) \cap \pi_0\mathrm{Q}^-(3, q^2)$ does not contain a line.

Then the $q^4 + 1$ lines through π_0 on $\mathrm{Q}^-(3, q^2)$ have at most $q + 1$ points of $\mathrm{H}(4, q^2)$. So

$$\begin{aligned} |\mathrm{H}(4, q^2) \cap \pi_0\mathrm{Q}^-(3, q^2)| &\leqslant (q + 1)(q^4 + 1) && (1)\\ &\leqslant q^5 + q^4 + q + 1. && (2)\end{aligned}$$

This upper bound is also determined in [5, Subsection 3.3.1].

Case II: $\mathrm{H}(4, q^2) \cap \pi_0\mathrm{Q}^-(3, q^2)$ contains at least one line.

Lemma 3.7 *If* $\mathrm{H}(4, q^2) \cap \pi_0\mathrm{Q}^-(3, q^2)$ *contains at least one line* L*, then* $\mathrm{H}(4, q^2) \cap \pi_0\mathrm{Q}^-(3, q^2)$ *contains at most* $2(q + 1)$ *lines.*

Proof. Since $L \subset \mathrm{H}(4, q^2) \cap \pi_0\mathrm{Q}^-(3, q^2)$, necessarily $\pi_0 \subset \mathrm{H}(4, q^2) \cap \pi_0\mathrm{Q}^-(3, q^2)$. Every line L' of $\mathrm{H}(4, q^2) \cap \pi_0\mathrm{Q}^-(3, q^2)$ passes through π_0, so lies in the tangent hyperplane $T_{\pi_0}(\mathrm{H}(4, q^2))$. This hyperplane intersects $\pi_0\mathrm{Q}^-(3, q^2)$ in a cone $\pi_0\mathrm{Q}(2, q^2)$ if there are at least two lines contained in $\mathrm{H}(4, q^2) \cap \pi_0\mathrm{Q}^-(3, q^2)$. Since $L \subset \mathrm{H}(4, q^2) \cap \pi_0\mathrm{Q}^-(3, q^2)$, it defines a point of $\mathrm{H}(2, q^2) \cap \mathrm{Q}(2, q^2)$, with $\mathrm{H}(2, q^2)$ and $\mathrm{Q}(2, q^2)$ the basis of the tangent cone $T_{\pi_0}(\mathrm{H}(4, q^2))$ and of $\pi_0\mathrm{Q}^-(3, q^2) \cap T_{\pi_0}(\mathrm{H}(4, q^2))$. By Bézout's theorem, $|\mathrm{H}(2, q^2) \cap \mathrm{Q}(2, q^2)| \leq 2(q + 1)$. So at most $2(q + 1)$ lines of $\pi_0\mathrm{Q}^-(3, q^2)$ lie completely on $\mathrm{H}(4, q^2)$. $\square$

By the previous lemma, we have:

$$\begin{aligned} |\mathrm{H}(4, q^2) \cap \pi_0\mathrm{Q}^-(3, q^2)| &\leqslant 2(q + 1)(q^2 + 1) + (q^4 - 2q - 1)(q + 1) && (3)\\ &\leqslant q^5 + q^4 + 2q^3 - q + 1. && (4)\end{aligned}$$

3.3 The quadric cone $\mathrm{Q} = \pi_0\mathrm{Q}^+(3, q^2)$

(see also [5, Section 3.1]) We can describe $\pi_0\mathrm{Q}^+(3, q^2)$ by $q^2 + 1$ planes defined by π_0 and the lines of one regulus of $\mathrm{Q}^+(3, q^2)$. No plane lies completely on $\mathrm{H}(4, q^2)$, so every plane shares at most $q^3 + q^2 + 1$ points, of a cone $P\mathrm{H}(1, q^2)$, with $\mathrm{H}(4, q^2)$. Hence,

$$\begin{aligned} |\mathrm{H}(4, q^2) \cap \pi_0\mathrm{Q}^+(3, q^2)| &\leqslant (q^2 + 1)(q^3 + q^2 + 1) && (5)\\ &\leqslant q^5 + q^4 + q^3 + 2q^2 + 1. && (6) \end{aligned}$$

3.4 The quadric cone $\mathrm{Q} = \pi_1\mathrm{Q}(2, q^2)$

(see also [5, Section 3.1]) Also this quadric can be described by $q^2 + 1$ planes, so as above

$$|\mathrm{H}(4, q^2) \cap \pi_1\mathrm{Q}(2, q^2)| \leqslant q^5 + q^4 + q^3 + 2q^2 + 1.$$

3.5 The quadric cone $\mathrm{Q} = \pi_2\mathrm{Q}^-(1, q^2)$

Then we have in fact the intersection of a plane with $\mathrm{H}(4, q^2)$. So this intersection size will be smaller than the previous bounds.

3.6 Conclusion

Let Q be a quadric in $\mathrm{PG}(4, q^2)$.

Theorem 3.8 *If* $|\mathrm{Q}\cap\mathrm{H}(4, q^2)| > q^5+3q^4+2q^2+q+1$, *then* Q *is the union of 2 hyperplanes.*

Proof. From Lemmata 3.3 and 3.6, we know that the intersection size of the non-singular quadric $\mathrm{Q}(4, q^2)$ with $\mathrm{H}(4, q^2)$ is at most $q^5+3q^4+2q^2+q+1$. For the different intersection sizes of other quadrics with $\mathrm{H}(4, q^2)$, (2), (4), and (6) learn us that they are smaller than the previous one. So this proves the theorem. □

From now on, we will denote this bound by $W_4 = q^5 + 3q^4 + 2q^2 + q + 1$.

4 General case

Let Q be a quadric in $\mathrm{PG}(N, q^2)$.

Theorem 4.1 *If* $|\mathrm{Q}\cap\mathrm{H}(N, q^2)| > (q^2+2)^{N-4}W_4$, *then* Q *is the union of two hyperplanes, for dimension* $N < O(q^2)$.

Proof. Part 1. Denote $(q^2 + 2)^{N-4}W_4$ by W_N. The bound is valid for $N = 4$ (Theorem 3.8).
Suppose that the lemma holds for dimension $N - 1$. By induction, we show that the bound is true for dimension N.

Select $(q^2+2)^{N-4}W_4$ points P of $\mathrm{Q} \cap \mathrm{H}(N,q^2)$ and count the incidences (P,H), with $P \in \mathrm{Q} \cap \mathrm{H}(N,q^2)$ and H a tangent hyperplane to $\mathrm{H}(N,q^2)$. This gives

$$((q^2+2)^{N-4}W_4)|P\mathrm{H}(N-2,q^2)| = |\mathrm{H}(N,q^2)|X_N,$$

with X_N the average number of those $(q^2+2)^{N-4}W_4$ points of $\mathrm{Q} \cap \mathrm{H}(N,q^2)$ in a tangent hyperplane to $\mathrm{H}(N,q^2)$.

So some tangent hyperplane $T_P(\mathrm{H}(N,q^2))$, $P \in \mathrm{H}(N,q^2)$, contains at most

$$\begin{aligned} X_N &\leqslant \frac{((q^2+2)^{N-4}W_4)((q^{N-1}+(-1)^{N-2})(q^{N-2}+(-1)^{N-1})q^2+q^2-1)}{(q^{N+1}+(-1)^N)(q^N+(-1)^{N+1})} \\ &\leqslant W_{N-1}(1+\frac{3}{q^2-1}), \end{aligned}$$

of those points.

There remain more than $(q^2+2)W_{N-1} - W_{N-1}(1+\frac{3}{q^2-1}) = (q^2+1-\frac{3}{q^2-1})W_{N-1}$ points in $\mathrm{Q} \cap \mathrm{H}(N,q^2)$, not lying in this tangent hyperplane $T_P(\mathrm{H}(N,q^2))$. Take an arbitrary $\mathrm{H}(N-3,q^2)$ on the base $\mathrm{H}(N-2,q^2)$ of $T_P(\mathrm{H}(N,q^2)) \cap \mathrm{H}(N,q^2)$. We do not know $|\mathrm{H}(N-3,q^2) \cap \mathrm{Q} \cap \mathrm{H}(N,q^2)|$, but we know that the q^2+1 hyperplanes through $\langle P, \mathrm{H}(N-3,q^2)\rangle$ are $T_P(\mathrm{H}(N,q^2))$, the only tangent hyperplane through $\langle P, \mathrm{H}(N-3,q^2)\rangle$, and q^2 hyperplanes intersecting $\mathrm{H}(N,q^2)$ in a non-singular Hermitian variety $\mathrm{H}(N-1,q^2)$.

So one of them, denoted by π, contains more than $\frac{(q^2+1-\frac{3}{q^2-1})W_{N-1}}{q^2} \geq W_{N-1}$ points of the intersection. Then in this hyperplane π, since $|\pi \cap \mathrm{Q} \cap \mathrm{H}(N-1,q^2)| > W_{N-1}$, $\pi \cap \mathrm{Q}$ is the union of two $(N-2)$-dimensional spaces.

Part 2. The only quadrics containing $(N-2)$-dimensional spaces are $\pi_{N-4}\mathrm{Q}^+(3,q^2)$, $\pi_{N-2}\mathrm{Q}^+(1,q^2)$, and $\pi_{N-3}\mathrm{Q}(2,q^2)$.

We want to eliminate the quadrics $\pi_{N-4}\mathrm{Q}^+(3,q^2)$ and $\pi_{N-3}\mathrm{Q}(2,q^2)$; they both can be described as the union of q^2+1 $(N-2)$-dimensional spaces π_{N-2}. The largest intersection of $\pi_{N-2} \cap \mathrm{H}(N,q^2)$ comes from a Hermitian variety which is the union of $q+1$ distinct $(N-3)$-dimensional spaces sharing an $(N-4)$-dimensional space and this has size

$$(q+1)q^{2N-6}+q^{2N-8}+\cdots+q^2+1 = q^{2N-5}+q^{2N-6}+q^{2N-8}+\cdots+q^2+1.$$

If this would be the case for all these q^2+1 distinct π_{N-2}, we would get at most an intersection size $(q^2+1)(q^{2N-5}+q^{2N-6}+q^{2N-8}+\cdots+q^2+1)$ of these quadrics with $\mathrm{H}(N,q^2)$. Since $(q^2+2)^{N-4}W_4 > (q^2+1)(q^{2N-5}+q^{2N-6}+q^{2N-8}+\cdots+q^2+1)$, these quadrics cannot occur.

So $\mathrm{Q} = \pi_{N-2}\mathrm{Q}^+(1,q^2)$ which is the union of two hyperplanes. $\square$

Remark 4.2 *The condition $N < O(q^2)$ arises from the fact that only for $N < O(q^2)$, the value $(q^2+2)^{N-4}W_4$ is smaller than or equal to the intersection size of two hyperplanes with a non-singular Hermitian variety $\mathrm{H}(N,q^2)$. Here, necessarily $N < q^2/3$.*

5 Structure of small weight codewords

We proved in Theorem 4.1 that the small weight codewords of $\mathrm{C}_2(\mathrm{X})$, X a non-singular Hermitian variety in $\mathrm{PG}(N, q^2)$, $O(q^2) > N \geq 4$, correspond to the intersections of X with the quadrics consisting of the union of two hyperplanes. We now count the number of codewords obtained via the intersections of X with the union of two hyperplanes.

Consider a quadric Q which is a union of two hyperplanes, then Q defines $q^2 - 1$ codewords of $\mathrm{C}_2(\mathrm{X})$, equal to each other up to a non-zero scalar multiple.

It could be that a quadric Q' which also is a union of two hyperplanes, but different from Q, defines the same $q^2 - 1$ codewords of $\mathrm{C}_2(\mathrm{X})$. However, this can be excluded for $N \geq 4$ in the following way.

If the quadric Q, which is the union of the two hyperplanes Π_1 and Π_2, and the quadric Q', which is the union of the two hyperplanes Π_1' and Π_2', define the same codewords of $\mathrm{C}_2(\mathrm{X})$, then $(\Pi_1 \cup \Pi_2) \cap \mathrm{X} = (\Pi_1' \cup \Pi_2') \cap \mathrm{X}$. Assume that $\Pi_1' \neq \Pi_1, \Pi_2$. Then the hyperplane intersection $\Pi_1' \cap \mathrm{X}$ must be contained in the two $(N-2)$-dimensional intersections $\Pi_1' \cap \Pi_1 \cap \mathrm{X}$ and $\Pi_1' \cap \Pi_2 \cap \mathrm{X}$. But the smallest possible intersection size of a hyperplane with X is larger than twice the largest possible intersection size of an $(N-2)$-dimensional space with X. So this case does not occur.

Hence, to calculate the number of codewords arising from the union of two hyperplanes, we simply check which unions of two hyperplanes determine codewords of a particular weight (Tables 1, 2 and 3); we then count how many such pairs of hyperplanes there are in $\mathrm{PG}(N, q^2)$, and then we multiply this number by $q^2 - 1$ since a union of two hyperplanes defines $q^2 - 1$ non-zero codewords which are a scalar multiple of each other. For $N \geq 4$, this determines the precise number of codewords of the smallest weights in $\mathrm{C}_2(\mathrm{X})$ (Table 3).

We determine the geometrical construction of the smallest weight codewords. They correspond to the intersection of $\mathrm{H}(N, q^2)$ with $\pi_{N-2}\mathrm{Q}^+(1, q^2)$. The quadric $\pi_{N-2}\mathrm{Q}^+(1, q^2)$ consists of two hyperplanes, which we will denote by Π_1 and Π_2, through an $(N-2)$-dimensional space π_{N-2}. We recall that a hyperplane intersects $\mathrm{H}(N, q^2)$ either in a non-singular Hermitian variety $\mathrm{H}(N-1, q^2)$ or, in case it is a tangent hyperplane, in a cone $\pi_0\mathrm{H}(N-2, q^2)$. This $(N-2)$-dimensional space π_{N-2} can intersect $\mathrm{H}(N, q^2)$ in different ways and this gives us different weight codewords. Starting from the intersection of $\pi_{N-2} \cap \mathrm{H}(N, q^2)$, we determine the different intersection sizes and small weights of $C_2(\mathrm{X})$.

For the intersection of π_{N-2} with $\mathrm{H}(N, q^2)$, there are three possibilities. This intersection is either a non-singular Hermitian variety $\mathrm{H}(N-2, q^2)$, a singular Hermitian variety $\pi_0\mathrm{H}(N-3, q^2)$ with vertex the point π_0 and base the non-singular Hermitian variety $\mathrm{H}(N-3, q^2)$, or a singular Hermitian variety $L\mathrm{H}(N-4, q^2)$ with vertex the line L and base the non-singular Hermitian variety $\mathrm{H}(N-4, q^2)$.

In Table 1, we denote the different possibilities for the intersection of $\mathrm{X} = \mathrm{H}(N, q^2)$ with the union of two hyperplanes Π_1 and Π_2.

		$\pi_{N-2} \cap \mathrm{H}(N, q^2)$	$\lvert \mathrm{X} \cap (\Pi_1 \cup \Pi_2) \rvert$
(1)	(1.1)	$\mathrm{H}(N-2, q^2)$	$2\lvert \mathrm{H}(N-1, q^2)\rvert - \lvert \mathrm{H}(N-2, q^2)\rvert$
	(1.2)	$\mathrm{H}(N-2, q^2)$	$\lvert \mathrm{H}(N-1, q^2)\rvert + \lvert \pi_0\mathrm{H}(N-2, q^2)\rvert - \lvert \mathrm{H}(N-2, q^2)\rvert$
	(1.3)	$\mathrm{H}(N-2, q^2)$	$2\lvert \pi_0\mathrm{H}(N-2, q^2)\rvert - \lvert \mathrm{H}(N-2, q^2)\rvert$
(2)	(2.1)	$\pi_0\mathrm{H}(N-3, q^2)$	$\lvert \mathrm{H}(N-1, q^2)\rvert + \lvert \pi_0\mathrm{H}(N-2, q^2)\rvert - \lvert \pi_0\mathrm{H}(N-3, q^2)\rvert$
	(2.2)	$\pi_0\mathrm{H}(N-3, q^2)$	$2\lvert \mathrm{H}(N-1, q^2)\rvert - \lvert \pi_0\mathrm{H}(N-3, q^2)\rvert$
(3)	(3.1)	$L\mathrm{H}(N-4, q^2)$	$2\lvert \pi_0\mathrm{H}(N-2, q^2)\rvert - \lvert L\mathrm{H}(N-4, q^2)\rvert$

Table 1

In the second table, we give the intersection sizes: we split the table up into the cases N even and N odd.

		$\lvert \mathrm{X} \cap (\Pi_1 \cup \Pi_2) \rvert$
(1)	(1.1)	$2q^{2N-3} + q^{2N-5} + q^{2N-7} + \cdots + q^{N+1} + q^{N-1} + 2q^{N-2} + q^{N-4} + \cdots + q^2 + 1$
	(1.2)	$2q^{2N-3} + q^{2N-5} + q^{2N-7} + \cdots + q^{N+1} + 2q^{N-2} + q^{N-4} + \cdots + q^2 + 1$
	(1.3)	$2q^{2N-3} + q^{2N-5} + q^{2N-7} + \cdots + q^{N+1} - q^{N-1} + 2q^{N-2} + q^{N-4} + \cdots + q^2 + 1$
(2)	(2.1)	$2q^{2N-3} + q^{2N-5} + q^{2N-7} + \cdots + q^{N+1} + q^{N-2} + q^{N-4} + \cdots + q^2 + 1$
	(2.2)	$2q^{2N-3} + q^{2N-5} + q^{2N-7} + \cdots + q^{N+1} + q^{N-1} + q^{N-2} + q^{N-4} + \cdots + q^2 + 1$
(3)	(3.1)	$2q^{2N-3} + q^{2N-5} + q^{2N-7} + \cdots + q^{N+1} + q^{N-2} + q^{N-4} + \cdots + q^2 + 1$

Table 2 (a): N even

		$\lvert \mathrm{X} \cap (\Pi_1 \cup \Pi_2) \rvert$
(1)	(1.1)	$2q^{2N-3} + q^{2N-5} + q^{2N-7} + \cdots + q^{N+2} + q^{N} - q^{N-2} + q^{N-3} + \cdots + q^2 + 1$
	(1.2)	$2q^{2N-3} + q^{2N-5} + q^{2N-7} + \cdots + q^{N} + q^{N-1} - q^{N-2} + q^{N-3} + \cdots + q^2 + 1$
	(1.3)	$2q^{2N-3} + q^{2N-5} + q^{2N-7} + \cdots + q^{N} + 2q^{N-1} - q^{N-2} + q^{N-3} + \cdots + q^2 + 1$
(2)	(2.1)	$2q^{2N-3} + q^{2N-5} + q^{2N-7} + \cdots + q^{N} + q^{N-1} + q^{N-3} + \cdots + q^2 + 1$
	(2.2)	$2q^{2N-3} + q^{2N-5} + q^{2N-7} + \cdots + q^{N} + q^{N-3} + \cdots + q^2 + 1$
(3)	(3.1)	$2q^{2N-3} + q^{2N-5} + q^{2N-7} + \cdots + q^{N} + q^{N-1} + q^{N-3} + \cdots + q^2 + 1$

Table 2 (b): N odd

From the intersection sizes listed in Table 2, we now determine the smallest weights for $C_2(\mathrm{X})$ by subtracting the size of the intersection $\mathrm{Q} \cap X$ from the length of the code $C_2(\mathrm{X})$. In the same table, we list the number of such codewords. We again split up the table into the cases N even and N odd.

	Weight	Number of codewords for $N \geq 4$
(1.1)	$w_1 = q^{N-2}(q^{N+1} - q^{N-1} - q - 1)$	$\frac{(q^{N+1}+1)(q^N-1)q^{2N-1}(q-1)(q^2-q-1)}{2(q+1)}$
(2.2)	$w_1 + q^{N-2}$	$\frac{(q^{N+1}+1)(q^N-1)q^N(q-1)(q^{N-1}+1)}{2}$
(1.2)	$w_1 + q^{N-1}$	$(q^{N+1}+1)(q^N-1)q^{2N-1}(q-1)$
(2.1)+(3.1)	$w_1 + q^{N-1} + q^{N-2}$	$\frac{(q^{N+1}+1)(q^N-1)q^N(q^{N-1}+1)}{q+1} + \frac{(q^{N+1}+1)(q^N-1)q^2(q^{N-1}+1)(q^{N-2}-1)}{2(q^4-1)}$
(1.3)	$w_1 + 2q^{N-1}$	$\frac{(q^{N+1}+1)(q^N-1)q^{2N-1}}{2}$

Table 3 (a): N even, $N < O(q^2)$

	Weight	Number of codewords for $N \geq 5$
(1.3)	$w_1 = q^{N-2}(q^{N+1} - q^{N-1} - q + 1)$	$\frac{(q^{N+1}-1)(q^N+1)q^{2N-1}}{2}$
(2.1)+(3.1)	$w_1 + q^{N-1} - q^{N-2}$	$\frac{(q^{N+1}-1)(q^N+1)q^N(q^{N-1}-1)}{q+1} + \frac{(q^{N+1}-1)(q^N+1)(q^{N-1}-1)(q^{N-2}+1)q^2}{2(q^2-1)}$
(1.2)	$w_1 + q^{N-1}$	$(q^{N+1}-1)(q^N+1)q^{2N-1}(q-1)$
(2.2)	$w_1 + 2q^{N-1} - q^{N-2}$	$\frac{(q^{N+1}-1)(q^N+1)q^N(q^{N-1}-1)(q-1)}{2}$
(1.1)	$w_1 + 2q^{N-1}$	$\frac{(q^{N+1}-1)(q^N+1)q^{2N-1}(q-1)(q^2-q-1)}{2(q+1)}$

Table 3 (b): N odd, $N < O(q^2)$

To conclude this article, we restate the conjecture of Edoukou [5] regarding the smallest weights of the functional codes $C_2(\mathrm{X})$, X a non-singular Hermitian variety of $\mathrm{PG}(N, q^2)$; a conjecture which we have proven to be true for small dimensions N.

Conjecture. The smallest weights of the functional codes $C_2(\mathrm{X})$, X a non-singular Hermitian variety of $\mathrm{PG}(N, q^2)$, arise from the quadrics Q which are the union of two hyperplanes of $\mathrm{PG}(N, q^2)$.

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